

On the effective character of a non abelian DBI action

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Abstract

We study the way Lorentz covariance can be reconstructed from Matrix Theory as a IMF description of M-theory. The problem is actually related to the interplay between a non abelian Dirac-Born-Infeld action and Super-Yang-Mills as its generalized non-relativistic approximation. All this physics shows up by means of an analysis of the asymptotic expansion of the Bessel functions K_ν that profusely appear in the computations of amplitudes at finite temperature and solitonic calculations. We hope this might help to better understand the issue of getting a Lorentz covariant formulation in relation with the $N \rightarrow +\infty$ limit. There are also some computations that could be of some interest in Relativistic Statistical Mechanics.

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1 Introduction

It is widely admitted that Matrix Theory is a non perturbative formulation of eleven dimensional M-theory either as its discrete light cone quantization (DLCQ) or in the infinite momentum frame (IMF) in the limit in which one connects with the perturbative superstring and its related objects as Dp-branes (see for example [1] and references therein).

What we call Matrix Theory is the Quantum Mechanics which results from the dimensional reduction of $\mathcal{N} = 1$ SYM theory in ten dimensions to $0 + 1$ dimensions. As a system in one dimension it has eight target degrees of freedom of bosonic and fermionic kind. With them one can finally describe the 256 ($=128+128$) degrees of freedom $\mathcal{N} = 1$ supergravity has in eleven dimensions. A light-cone and an IMF description of a theory in D dimensions is actually a theory in $D - 2$ dimensions. Eight are the number of degrees of freedom that result from the gauge symmetries the SYM has. In the Hamiltonian formalism of this quantum mechanics, it is easy to see that the extra Gauss constraints that reduce the number of SYM gauge fields to eight become trivial when the gauge group becomes abelian as actually happens in the limit in which the N branes get apart. This is the free limit which corresponds to the low energy limit of M-theory in which massless $\mathcal{N} = 1$ SUGRA in eleven dimensions is recovered. And so one finally finds oneself with nine degrees of freedom which are the number of transverse dimensions in a eleven dimensional space-time.

If from the very beginning one puts the system at finite temperature, the DLCQ of M-theory that Matrix Theory provides easily produces a canonical free energy that, in the classical limit when the radius R_+ of x^- goes to infinity, gives the Helmholtz free energy of the corresponding massless SUGRA [2]. This is an expected result because a generic observer is linked to the light-cone frame by a (limiting) Lorentz transformation [3].

What we will do is to study the relationship between the IMF calculation of the free energy and the same magnitude gotten for massless $\mathcal{N} = 1$ SUGRA in a generic Lorentz frame (what loosely could be called the 'Lorentz covariant' version of that amplitude). This will reveal us the way IMF computations can be connected with generic frame descriptions. In particular, we will learn, in connection with the Matrix Theory conjecture, on the physical meaning of the Bessel functions K_ν which profusely appear in finite temperature and solitonic calculations (see [4] for the special case of D-instanton calculations) and, in particular, their asymptotic expansions on the variable βm . We will actually see that such expansion is one coming from the Galilean average of the relativistic corrections of the energy of the KK modes in the uncompactified ten dimensional space-time. This way we will show that the expansion directly results from one for the exponential of the relativistic energy of a Kaluza-Klein-particle of mass m_k in ten dimensions as a power series in $(v_T/c)^2$ where c is the speed of light and v_T the modulus of the nine dimensional transverse velocity. The asymptotic character will be a result of Watson's Lemma because taking the trace over the nine open dimensions (which amounts in fact to a Galilean average) will be equivalent to making a Laplace transform of the power series in $(v_T/c)^2$. The

IMF limit keeps, from this series, the rest energy plus the Galilean contribution. This tells us that there must be a difference in the way we catch the eleven dimensional degrees of freedom if we compare the IMF approach and the DLCQ picture. In particular, in the IMF picture it will be relevant whether those degrees of freedom can be captured without an infinite number of anti-D0-branes. The anti-D0-branes are explicit in the generic description in which negative values of p_{11} appear. We will actually show how all the degrees of freedom of the uncompactified SUGRA can be gotten by using only positively RR-charged states. After all, DLCQ and the IMF description have to coincide with any generic description when we take the decompactification limit [5].

From here, it will be easy to construct the effective Lagrangian of the (free) single-object as seen by a generic observer. That is to say, the D-particle description of the classical limit of M-theory. It will be none other than the Dirac-Born-Infeld (DBI for short) Lagrangian for $U(1)^k$. It would be a limit of the one for $U(k)$ defined as some still unknown generalization of the abelian DBI Lagrangian. It is worth remarking that the trace would always be performed through an expansion in powers of matrices taken as a definition for the square root of a non diagonal matrix. That would finally produce an asymptotic series for the partition function after making an expansion for the exponential. In other words, the effective character of any non abelian generalization of the DBI action would show in which, through Watson's Lemma, one would get an asymptotic expansion for the canonical free energy. We will see that to actually get the complete number of degrees of freedom after decompactifying to eleven dimensions, a more subtle view on the extension through analytic continuation of the effective character of this expansion is needed.

First of all, let us recall in section two the finite temperature classical limit of M-theory as seen in a generic Lorentz frame. After that, we will present the same problem from the IMF view. Section four will explain how to relate the asymptotic expansion with any eventual generalization of the DBI action for the non abelian $U(k)$ case and will establish the conclusions too.

2 The free energy as seen by a generic observer

The single-object contribution to the canonical partition function for $\mathcal{N} = 1$, $d = 11$ SUGRA on $S^1 \times \mathbb{R}^{10}$ as calculated in a generic frame admits a proper time representation which reads

$$Z_1[\beta] = -\beta F_{MB}[\beta] = 256 V_{10} \beta (2\pi)^{-5} \int_0^{+\infty} ds s^{-13/2} \theta_3 \left[0, \frac{i(2\pi R_{11})^2}{2\pi s} \right] e^{-\beta^2/2s} \quad (2.1)$$

where F_{MB} stands for the Maxwell-Boltzmann contribution to the Helmholtz free energy.

As the classical limit of M-theory on $\mathbb{R}^{10} \times S^1$, this is the result of taking the limit $\sqrt{\alpha'} \ll (\sqrt{\alpha'} g_s)$ with α' going to zero and $R_{11} = g_s \sqrt{\alpha'}$, cf. [6]. This becomes more explicit

after performing a Poisson re-summation over the Jacobi θ_3 function to see physics as pictured in ten dimensions, namely

$$Z_1[\beta] = 256V_9\beta(2\pi)^{-5} \int_0^{+\infty} ds s^{-6} \left(2 \sum_{k=1}^{+\infty} e^{-m_k^2 s/2} e^{-\beta^2/2s} + e^{-\beta^2/2s} \right) \quad (2.2)$$

where the last term in the integrand gives the $\alpha' \rightarrow 0$ limit of the contribution of the free energy of the SST IIA which corresponds to the zero mode along the compact eleventh dimension. The sum over KK modes has been trailed to a sum on $m_k = k/R_{11}$ from one to infinity picking up a factor of two. This means that D0-branes (positive momentum) and anti-D0-branes (negative momentum) contribute just the same.

By performing the proper time integral one obtains

$$Z_1[\beta] = (k=0 \text{ contribution}) + 2\beta^{-4}V_9 \sum_{k=1}^{+\infty} 2(2\pi)^{-5} m_k^5 K_5[\beta m_k] \quad (2.3)$$

Here, K_5 is a modified Bessel function. It admits an asymptotic expansion which is

$$K_5[\beta m_k] = \sqrt{\frac{\pi}{2\beta m_k}} e^{-\beta m_k} \sum_{n=0}^{+\infty} \frac{\Gamma[n+11/2]}{\Gamma[n+1]\Gamma[-n+11/2]} (2\beta m_k)^{-n} \quad (2.4)$$

Let us now see the IMF description we get for this limit of M-theory. In there, it is assumed that Super-Yang-Mills is the Galilean approximation of a Lagrangian that covariantly would read as a non abelian generalization of the DBI action.

3 Finite temperature (low energy) M-theory as seen in the IMF

An infinite momentum frame description of M-theory is one in which the total momentum in the longitudinal direction is much larger than all transverse momenta. The connection between the IMF frame and the traditional light-cone frame is provided by the fact that the momentum in the longitudinal direction on which the observer is boosted satisfies $p_L \gg p_-$. The M(atrix) theory proposal in its IMF version assumes that a covariant version would be achieved after adding up the relativistic corrections to the Galilean part.

The connection between the IMF description and that in a generic Lorentz frame is based on the fact that in the IMF limit where $p_{11} \gg p_T$ we have

$$p^0 = \sqrt{p_{11}^2 + \vec{p}_T^2} = |p_{11}| \left\{ 1 + \frac{p_T^2}{2|p_{11}|^2} + \mathcal{O}\left[\left(\frac{p_T^2}{p_{11}^2}\right)^2\right] \right\} \quad (3.5)$$

This is the expression of the energy of a free massless particle in eleven dimensions. This will be the Hamiltonian of the relativistic Supersymmetric Quantum Mechanics of the

KK-modes with mass $|p_{11}|$ in ten dimensions, i.e., the Hamiltonian for a free bound state of D0-branes. Now, since x^{11} is compact, p_{11} will be quantized in units of $1/R_{11}$.

Nonetheless, what we want to do is to compute $Z_1(\beta)$ from the IMF by introducing the relativistic corrections. Instead of using the expression of the energy as the square root, we write p^0 as the IMF approximation plus the corrections. Restoring the speed of light c for the moment, and writing m_k for the mass of the KK mode, $m_k = \frac{k}{cR_{11}}$, we find

$$\frac{p^0}{m_k c} = 1 + \frac{p_T^2}{2m_k^2 c^2} + \mathcal{C}\left(\frac{p_T^2}{m_k^2 c^2}\right) \quad (3.6)$$

where $\mathcal{C}[x]$ can be represented by the series

$$\mathcal{C}[x] = (1/2)! \sum_{n=1}^{\infty} \frac{x^{n+1}}{(-1/2 - n)!(n+1)!} \quad (3.7)$$

that, to directly compute several orders, can be used as a fast way of reading the n -th derivative of $\mathcal{C}[x]$ at $x = 0$.

All together will provide us with an expansion in powers of $p_T^2/(m^2 c^2)$ for $e^{-\beta c p^0}$ that is²,

$$e^{-\beta c p^0} = e^{-\beta m_k c^2} e^{-\beta p_T^2/2m_k} \sum_{n=0}^{+\infty} a_n [\beta m_k c^2] \left(\frac{p_T^2}{m_k^2 c^2}\right)^n \quad (3.8)$$

where

$$a_n[y] = \frac{1}{n!} \frac{d^n}{dx^n} e^{-y \mathcal{C}[x]} \Big|_{x=0} \quad (3.9)$$

will only depend on dimensionful quantities through the dimensionless combination $y \equiv \beta m_k c^2$ and are, in fact, finite degree polynomials on this variable. It is remarkable that we are keeping apart the Boltzmann weight corresponding to the rest plus the Galilean kinetic energy. This factorization will actually be taken as the IMF way of computing the partition function. For instance, neglecting terms of $\mathcal{O}(v_T^{10}/c^{10})$ we have

$$\begin{aligned} e^{-\beta c p^0} &= e^{-\beta m_k c^2} e^{-\beta p_T^2/2m_k} \left\{ 1 + \frac{1}{8}(\beta m_k c^2) \left(\frac{p_T}{m_k c}\right)^4 - \frac{1}{16}(\beta m_k c^2) \left(\frac{p_T}{m_k c}\right)^6 \right. \\ &\quad \left. + \frac{1}{128}(\beta m_k c^2)(\beta m_k c^2 + 5) \left(\frac{p_T}{m_k c}\right)^8 + \mathcal{O}\left[\left(\frac{p_T}{m_k c}\right)^{10}\right] \right\} \end{aligned} \quad (3.10)$$

We can now compute the partition function by taking the trace over transverse momenta and summing over k as well. For the transverse part we find from (3.8)

$$Z_T[\beta] = \text{Tr}_{\vec{p}_T} e^{-\beta c p^0} = e^{-\beta m_k c^2} \sum_{n=0}^{+\infty} a_n [\beta m_k c^2] \left\langle \left(\frac{p_T}{m_k c}\right)^{2n} \right\rangle_{\beta} \quad (3.11)$$

²We would like to emphasize the fact that it is an expansion for the exponential of the energy times the inverse temperature and not necessarily for the square root

where $\langle \dots \rangle_\beta$ denotes the thermal average with respect to the Galilean measure; i.e., $\langle f[\vec{p}_T] \rangle_\beta = \int d\vec{p}_T f[\vec{p}_T] \exp(-\beta p_T^2/2m_k)$. Actually, we can try to guess what kind of series we can expect after thermal averaging. It can be easily checked that for a Galilean particle

$$\left\langle \left(\frac{p_T}{m_k c} \right)^{2n} \right\rangle_\beta = (-1)^n \left(\frac{2}{m_k c^2} \right)^n \frac{d^n}{d\beta^n} \langle 1 \rangle_\beta \sim (\beta m_k c^2)^{-n} \langle 1 \rangle_\beta \quad (3.12)$$

so in the end we will get a series in inverse powers of $\beta m_k c^2$. Notice that although the coefficients of each term in the expansion (3.8) are polynomials in this same argument, the order of these polynomials is always bounded by the corresponding power of $p_T/m_k c$ and thus we are only left with negative powers of $\beta m_k c^2$. The resulting expansion will then be reliable when $T \ll m_k c^2$ or, in other words, in the limit of large masses; this is what one physically should expect, since we started with a non-relativistic approximation.

Actually, we shall show that each term in the expansion can be exactly computed, and after tracing over the longitudinal discrete momentum, it can be found

$$\begin{aligned} Z_1[\beta] &= 256 \frac{V_9}{(2\pi)^9} \sum_{k=1}^{+\infty} e^{-\beta m_k c^2} \left(\frac{2\pi m_k}{\beta} \right)^{9/2} \times \\ &\times \sum_{n=0}^{+\infty} \frac{\Gamma[n + 11/2]}{\Gamma[n + 1] \Gamma[-n + 11/2]} (2\beta m_k c^2)^{-n} \end{aligned} \quad (3.13)$$

Incidentally, let us mention that it is precisely through this asymptotic expansion that we see, after identifying R_{11} with $g_s \sqrt{\alpha'}$, the expected string non-perturbative effects as seen in a light-cone or infinite momentum frame description, i.e., weighted by the exponential of minus a constant over a single power of the string coupling. Looking at the series we see that the $n = 0$ term closely resembles the full low energy DLCQ calculation (see [2]). Here, the term to the 9/2 power comes from the Galilean kinetic energy of a free particle of mass m_k in ten dimensions after integrating over the nine-dimensional (transverse, in eleven dimensions) momentum. The exponential is the contribution of the rest energy of this particle of mass m_k . If one naively and erroneously identified R_+ and R_{11} the difference between the $n = 0$ term in eq.(3.13) and the whole DLCQ calculation would be in the counting of the rest energy. It is also important to notice that in the trace over p_{11} we have restricted the summation over states with $k \geq 1$ because we assume that the system has been boosted along the longitudinal direction to make p_{11} positive. Since in the IMF we have that $p_{11} \gg p_T$ this means that we will have to take $R_{11} \rightarrow 0$ so the mass of the Kaluza-Klein states will be very large for a finite k . This is in agreement with the fact pointed out above in the sense that (3.13) is a large mass $m_k c^2 \gg T$ (or low transverse velocity $v_T \ll c$) expansion which is an asymptotic one over the index n and convergent over k for fixed n . One might expect that taking the $R_{11} \rightarrow \infty$ and then summing up on n does not have to give the same result as summing up the asymptotic expansion and finally taking the decompactification limit. Furthermore because, as we will see, different analytical continuations are involved in each limit.

If we compare with eq.(2.3) and eq.(2.4), we notice that the sum over n reconstructs a Bessel function of $\beta m_k c^2$ for a given k . The result is that the single-object partition function can be written as

$$Z_1[\beta] = 2 \frac{256}{\beta^9} \frac{V_9}{(2\pi)^5} \sum_{k=1}^{\infty} (\beta m_k c^2)^5 K_5[\beta m_k c^2] \quad (3.14)$$

where $K_\nu[z]$ are the modified Bessel functions. In this expression of $Z_1[\beta]$ we can relax the condition of small R_{11} since the series will be convergent for all values of the radius due to the exponential suppression for large k . If we take the limit of large radius in eq. (3.14) we find that the sum can be converted into an integral to give (setting again $c = 1$)

$$\lim_{R_{11} \rightarrow \infty} Z_1[\beta] = 2 \frac{256}{\beta^{10}} \frac{V_{10}}{(2\pi)^6} \int_0^\infty dx x^5 K_5[x] = 256 \frac{\Gamma[11/2]}{2\pi^{\frac{11}{2}}} V_{10} \beta^{-10} \quad (3.15)$$

By simply keeping the zero mode term in the Jacobi θ_3 in eq.(2.1) and performing the integral, it is easy to check that this result corresponds to one-half the value of the single-object partition function of a supergraviton in uncompactified eleven-dimensional space-time. This factor of two difference can be traced back to the fact that we are just summing over positive momenta in the x^{11} direction, which corresponds in the computation of the trace to integrate only over momenta lying on a hemisphere (i.e. restricting the polar angle on \mathbf{S}^9 to the interval $(0, \frac{\pi}{2})$). So, we miss half of the degrees of freedom of eleven dimensional SUGRA. After comparing with eq.(2.2) it is also obvious that this factor represents the anti-D0-brane contribution encoded in the negative KK momentum modes.

From a mathematical point of view and because of its Borel summability, summing up the asymptotic series would correspond to an analytical continuation through the Borel transformed series.

The other way around, if we keep n fixed and so take the decompactification limit then we trail the sum over k by an integral for a given n . More explicitly

$$\lim_{R_{11} \rightarrow +\infty} \sum_{k=1}^{+\infty} m_k^{9/2-n} e^{-\beta m_k} \longrightarrow \frac{(2\pi R_{11})}{(2\pi)} \int_0^{+\infty} dy y^{9/2-n} e^{-\beta y} \quad (3.16)$$

to give the following convergent (!) expansion on n

$$Z_1[\beta] = 256 \frac{V_9(2\pi R_{11})}{(2\pi)^{11/2}} \beta^{-10} \sum_{n=0}^{+\infty} \frac{\Gamma[n + 11/2]}{2^n \Gamma[n + 1]} \quad (3.17)$$

The convergent sum is easily proven to give $2^{11/2} \Gamma[11/2]$. Therefore now we get twice the result in eq.(3.15) or, in other words, the complete supergraviton. Physically this limit corresponds to decompactifying the Galilean average of each relativistic correction in order to finally sum up the outputs. Mathematically, the relevant issue is the need for an analytical continuation for $n \geq 6$ (n is the index that labeled the asymptotic series and so is zero or a positive integer) of the integral to which the k -sum goes because

of the well known restriction upon the standard integral representation of the gamma function. This is the key point to understand the different output when computing the decompactification limit before or after summing up the asymptotic expansion.

The convergent series we arrive at admits a simple physical interpretation if we substitute the integral representation for $\Gamma[n + 11/2]$ to finally write down a expression in non discrete LCF variables as

$$\frac{V_{10}}{(2\pi)^{10}} \int_0^{+\infty} dp^+ \int d\vec{p}_T e^{-\beta p^+} e^{\beta(p^+/2 - \vec{p}_T^2/2p^+)} \quad (3.18)$$

now the transverse momentum is a nine-dimensional vector.

This results from writing down the energy p^0 as $p^+ - p_L$ with $p_L = p^+/2 - p^-/2 = p^+/2 - \vec{p}_T^2/2p^+$. The convergent series in n comes out as a series for small p^+ representing the longitudinal contribution given by

$$Z_L[\beta, p^+] = \frac{V_9}{(2\pi)^9} e^{\beta p^+/2} \int d\vec{p}_T e^{-\beta \vec{p}_T^2/2p^+} \quad (3.19)$$

Here one must notice that this is the longitudinal contribution in the sense of the exact light cone frame.

Through the relationship $s = \beta/p^+$ amongst p^+ , β and the proper time s , this corresponds to an infrared expansion, i.e., one for large proper times. It is worth remarking that the zero term in this series coincides with what could be gotten by taking a large longitudinal momentum and then approaching p^+ by $2p_L$. In fact, this convergent series and the asymptotic one that one gets in eleven dimensions for large longitudinal momentum (the continuous IMF computation) run very close up to the fifth term.

Because of its crucial role, it is then important to show that there exists a way of getting the asymptotic expansion for the Bessel function as a result of Watson's Lemma. In the appendix we show how this actually works.

It is through the asymptotic series that one captures all the degrees of freedom in eleven dimensions. Physically, the starting point is an still unknown non abelian generalization of the DBI action. One thinks of this action written as a certain trace over a square root of a non diagonal matrix. This is a formal writing of the action dictated by a way of making explicit the ten dimensional relativistic character the action should have and its scalar character through a certain variation of a trace. In fact, what we are writing is a Lagrangian with an infinite number of terms are believed might sum up as a kind of a trace over a square root. The expansion with an infinite number of terms results as the way of defining the square root of a non diagonal matrix. Using the corresponding Hamiltonian given by a series with a finite radius of convergence to compute amplitudes like the free energy leads to computing momentum integrals which take the momentum to infinity and then away from the convergence region of the series. These integrals would finally be (here, we see this in the classical M-theory limit) Laplace transforms and, as a result of Watson's Lemma, would give an asymptotic series. This is the way a non abelian DBI action would work as an effective action and produce asymptotic series and it is in

there that the effective character is encoded. We want to make clear to the reader that we are not going to propose any modification to any non abelian generalization of the DBI action. More than that, we are trying to go further saying that irrespective of the precise form of such generalization, whenever this is given by an expansion with a finite radius of convergence, the description after decompactification one can get for M-theory includes an analytical continuation performed through the partition function represented as an asymptotic series.

4 The effective Lagrangian of the IMF calculation and the Dirac-Born-Infeld action

Recovering the asymptotic expansion of the Bessel function from the IMF description illustrates more than the way the decompactification limit should be taken and then eleven dimensional Lorentz covariance would be achieved. It is clear that the expansion of p^0 in relativistic corrections should be related to an expansion over the Dirac-Born-Infeld action for D0-branes. In the supersymmetric case, the generalization of this action to non abelian theories seems to involve the Lagrangian [7]

$$L = -\frac{c_0}{g_s} \text{Str} \sqrt{\det(\eta_{\mu\nu} - F_{\mu\nu})} \quad (4.20)$$

together with corrections that cannot be summed up as a symmetric trace (see [8]). Here $\eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$ is the flat Minkowski metric in the ten dimensional space where the Dp-brane lives and the field strength only depends on the first $p+1$ coordinates. The determinant is understood on the matrix character given by the space-time indices $\mu\nu$. The symmetric trace acts on the $U(N)$ indices and will be understood by expanding the square root in a multiple power series. Finally c_0 is a normalization constant we take for convenience and g_s is the string coupling constant.

In the case in which $p = 0$ and the group is truncated to $U(1)^N$ everything gets easier and one has

$$L = -\frac{c_0}{g_s} \text{Tr} \sqrt{\left(I - \sum_{i=1}^9 \left(\frac{dX_i}{dt} \right)^2 \right)} \quad (4.21)$$

with the gauge election $X_0 = A_0 = 0$, I being the $N \times N$ identity and X_i is a diagonal $N \times N$ matrix for every i from one to nine. The standard trace Tr coincides with the symmetric trace when matrices commute and the corrections to the symmetric trace should also disappear.

Let us now suppose that k elements in the diagonal of X_i coincide for all t . After tracing over, they will sum up to contribute

$$L_k = -\frac{c_0 k}{g_s} \sqrt{1 - \sum_{i=1}^9 \left(\frac{dx_i}{dt} \right)^2} \quad (4.22)$$

This is the way we represent the contribution of a bound state of k D0-branes. We also made the important extra assumption that there is only one of such states for each k at zero temperature. These are our KK modes in ten dimensions. To see how to characterize them, let us go to the Hamiltonian formalism by making a Legendre transform to get

$$H_k = \sqrt{m_k^2 + p_T^2}, \quad (4.23)$$

where we have already set the constant c_0 such that $c_0 k / g_s = k / R_{11} = m_k$.

The eigenstates of this Hamiltonian will be those of the transverse (in eleven dimensions) momentum and will also include k as a label of the particle to give its mass. It is worth to notice that because of its origin this is a positive number.

Computing its single particle partition function $Z_{1,k}$ is a very easy task. After that it is clear that to get $Z_1(\beta)$ one has to look at k as a quantum number and tracing over it. This clearly enhances the number of degrees of freedom because it is equivalent to assuming that, at finite temperature, there is no conservation of the RR charge. Another untied question is that of factor 256. In the single particle calculation in the Hamiltonian formalism it is another quantum number to trace over to give 2^8 since for a single particle at finite temperature, there is no difference between being a fermion or a boson. It is the spin quantum number of the quantum mechanics we call Matrix Theory.

With the necessary sum over k from one to infinity, one would get for $Z_1(\beta)$ the result in eq.(3.14) which gives the 'bad' decompactification limit. What happens is that we should start from the theory in interaction (with group $U(k)$) and then, before performing any kind of trace, one has to use an expansion for the square root what, by the way, is a tedious task even for the simple case $k = 2$. This expansion includes the rest energy plus the $U(k)$ Super-Yang-Mills contribution to the Hamiltonian and an infinite number of corrections. The contribution of this corrections averaged with the Galilean Super-Yang-Mills weight will finally produce an asymptotic series because of Watson's Lemma. The abelian limit of this series will be given by eq.(3.13) through the asymptotic series of a Bessel function interpreted as an IMF average of the relativistic corrections.

The key point is then to explain why, in the case of free D0-branes (the abelian case), we should prefer the asymptotic series expansion to the possibility of getting a result without using an expansion for the square root. The answer arises if one realizes that the abelian $U(1)^k$ case, from the M-theory point of view, must be understood as a limiting situation on the non abelian $U(k)$ picture. So, the way of defining a square root of a non diagonal matrix is by using an infinite expansion that however has a finite radius of convergence. Furthermore, trying to write down the adequate generalization of the DBI Hamiltonian for the non abelian case as some kind of trace of a square root might simply be a formal task enforced by trying to show the ten dimensional Lorentz invariance by means of a scalar gotten from a square root Hamiltonian. The infinite series, when computing amplitudes, will always give asymptotic expansions by Watson's Lemma. After all, this would only be an effective action with an infinite number of terms.

We then conjecture that in the $U(N)$ situation one always gets an asymptotic expansion that is the relevant mean to get the physics, in particular the decompactification limit

will be achieved by decompactifying the contribution of each term and making an analytic continuation after a particular order. This order, in the classical limit (free D0-branes in the Matrix picture), is given by the number of uncompactified dimensions as the integer part of $(D + 1)/2$ with D the dimension of the open space. It could be that D0-brane interactions moved this number down. It is obvious that compactifying dimensions lowers the order from which the integral representation of the gamma function fails. To us, the analytical continuation of the integral that represents the discrete sum for the big N limit means that the regularization of M-theory Matrix theory provides includes M-theory effects in the continuous limit (the anti-D0-branes in our computation) only after a kind of extra regularization over the amplitudes.

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Appendix

That the asymptotic expansion for the K_5 Bessel function can be obtained by an application of Watson's Lemma seems to be a trivial fact because the asymptotic series in negative powers of $\beta m_k c^2$ with coefficient

$$c_n = \frac{\Gamma[n + 11/2]}{2^n \Gamma[-n + 11/2] \Gamma[n + 1]} \quad (\text{A.24})$$

is Borel summable and Watson's Lemma is what is behind Borel summability.

However this is not the usual procedure in the mathematical literature to find the expansion of the modified Bessel functions for large argument. It seems that this could have been so because things do not appear that trivial after a first computation since by using the expansion for p^0 in powers of $|p_T/m_k c|^2$ (which converges whenever $|p_T/m_k c| < 1$) and averaging with the Galilean Boltzmann factor over \vec{p}_T the expansion of $e^{-\beta c p^0}$ in such powers produces for the single partition function per degree of freedom

$$\frac{Z_1[\beta]}{256} = \frac{V_9}{(2\pi)^9} \frac{\pi^{9/2}}{\Gamma[9/2]} \sum_{k=1}^{+\infty} e^{-\beta m_k c^2} \left(\frac{2m_k}{\beta} \right)^{9/2} \sum_{n=0}^{+\infty} 2^n a_n [\beta m_k c^2] \Gamma[n + 9/2] (\beta m_k c^2)^{-n} \quad (\text{A.25})$$

where a_n given in eq.(3.9) is a polynomial on the variable $\beta m_k c^2$.

In the process of getting eq.(A.25) one sees that integrating over \vec{p}_T to perform the Galilean average of the relativistic corrections is finally equivalent to making a Laplace transform. Here we see the gamma function provided by Watson's Lemma. However we are yet not done because the coefficients a_n are finite degree polynomials on the variable $y = \beta m_k c^2$ we must find. If we write them as

$$a_n [y] = \sum_a b_{n,a} y^a \quad (\text{A.26})$$

one can finally write $Z_1 [\beta]$ per degree of freedom as

$$\frac{Z_1[\beta]}{256} = \frac{V_9}{(2\pi)^9} \sum_{k=1}^{+\infty} e^{-\beta m_k c^2} \left(\frac{2\pi m_k}{\beta} \right)^{9/2} \sum_{n=0}^{+\infty} c_n (\beta m_k c^2)^{-n} \quad (\text{A.27})$$

with

$$c_n = \sum_{i=0}^{\infty} 2^{n+i} \frac{\Gamma[n+i+9/2]}{\Gamma[9/2]} b_{n+i,i} \quad (\text{A.28})$$

To get the coefficients $b_{n+i,i}$ defined in (A.26) we simply rewrite

$$e^{-y\mathcal{C}(x)} = e^y e^{yx/2} e^{-y\sqrt{1+x}} \quad (\text{A.29})$$

in terms of power series in the dummy variable x (for $x < 1$) and the variable y to obtain

$$b_{n+i,i} = \sum_{z=0}^i \sum_{u=0}^z \frac{(-1)^u}{(z-u)!(i-z)! u! 2^{i-z}} \binom{u/2}{n+z} \quad (\text{A.30})$$

For integer z , i and m only the odd values of u contribute to the first sum that can actually be taken up to infinity and summed up. This gives, after performing the sum over z (generalized hypergeometric functions help much to perform this task),

$$b_{n+i,i} = -\frac{\Gamma[i-n]}{2^{i+2n} \Gamma[i] \Gamma[1-2n] \Gamma[1+i+n]} \quad (\text{A.31})$$

From here one can easily see that $b_{2n+k,n+k} = 0$ with positive integer k showing that the sum over the index i in eq.(A.28) actually stops at n . There is no problem with setting i to zero, because $b_{n,0} = 0$ for $n \neq 0$ and $b_{0,0} = 1$ and the sum over i will start from $i = 1$ for $n > 0$. After substituting this into eq.(A.28) one finally gets the expected result for the coefficient c_n .

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